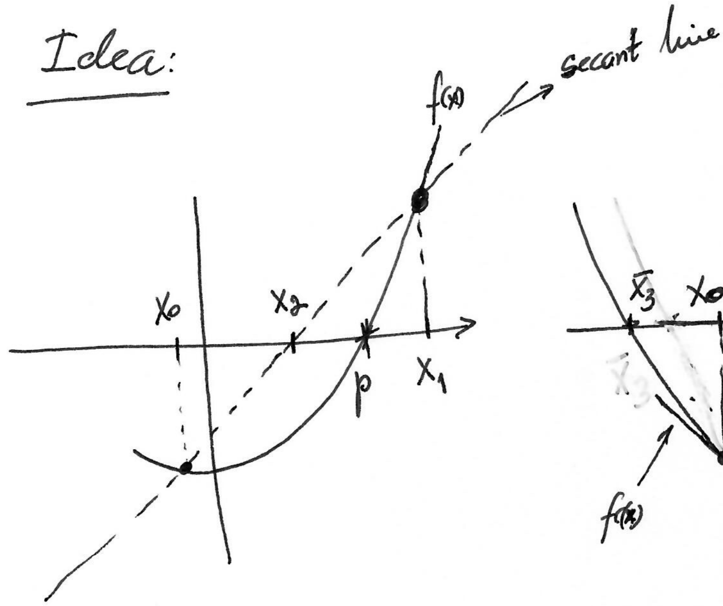
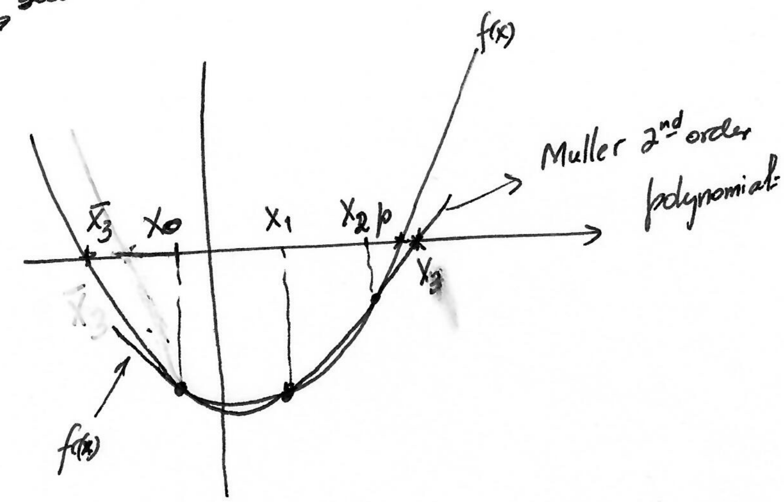
2.6 Zeros of Polynomials.

Idea:



Secant method



Muller's method.

Secant method consists of (a) choosing two estimates x_0 and x_1 of the root p.

(b) Construct a secant line through points: $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

(c) Find the intersection of the secant line with the x-axis. If we call this intersection x_2. Then, x_2 will be taken as a new approx. to p.

(d) steps (b) and (c) are repeated for $(x_1, f(x_1))$, $(x_2, f(x_2))$ to generate x_3, and so on.

Müller's Method. It Consists of:

a) Choosing three estimates: x_0, x_1, x_2 for the root p.

b) Constructing a parabola through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$, given by

$$\boxed{P(x) = a(x-x_2)^2 + b(x-x_2) + c} \qquad (2.1)$$

c) Finding a root, x_3 of (2.1), by finding (analytically) a root z_3 of

$$\boxed{P(z) = az^2 + bz + c} \qquad (2.2)$$

and solving $z_3 = x_3 - x_2$. i.e., $x_3 = z_3 + x_2$. x_3 is chosen such that $|z_3| = |x_3 - x_2|$ is the smallest. betw. two possibilities at most. (the two roots).

d) (b) and (c) are repeated with the new set of points: $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$. Thus, a new approx. x_4 is generated.

e) steps (b) and (c) are repeated until two consecutive approx. differs in a prescribed tolerance. It means

$$|z_n| = |x_n - x_{n-1}| < Tol$$

In the equation for the parabola $P(x)$, (2.1) a, b, and c are determined using the conditions that the parabola passes through the three points: $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$. This leads to the following three equations for the unknowns a, b, and c.

$$\begin{cases} a(x_0 - x_2)^2 + b(x_0 - x_2) + c = f(x_0) \\ a(x_1 - x_2)^2 + b(x_1 - x_2) + c = f(x_1) \\ a\cancelto{0}{(x_2 - x_2)}^2 + b\cancelto{0}{(x_2 - x_2)} + c = f(x_2) \end{cases}$$

→ Show that the solutions for this system are

$$c = f(x_2)$$

$$b = \frac{(x_0 - x_2)^2\left[f(x_1) - f(x_2)\right] - (x_1 - x_2)^2\left[f(x_0) - f(x_2)\right]}{(x_0 - x_2)(x_1 - x_2)(x_0 - x_1)}$$

$$a = \frac{(x_1 - x_2)\left[f(x_0) - f(x_2)\right] - (x_0 - x_2)\left[f(x_1) - f(x_2)\right]}{(x_0 - x_2)(x_1 - x_2)(x_0 - x_1)}$$

In fact, the solutions "a" and "b" for the system are

$$\begin{cases} a\,(x_0-x_2)^2 + b\,(x_0-x_2) = f(x_0) - f(x_2) \\ a\,(x_1-x_2)^2 + b\,(x_1-x_2) = f(x_1) - f(x_2) \end{cases}$$

Using Kramer's rule

$$a = \frac{\big(f(x_0)-f(x_2)\big)(x_1-x_2) - (x_0-x_2)\big(f(x_1)-f(x_2)\big)}{\underbrace{(x_0-x_2)^2(x_1-x_2) - (x_0-x_2)(x_1-x_2)^2}}$$

$$\to (x_0-x_2)\Big[(x_0-x_2)(x_1-x_2) - (x_1-x_2)^2\Big]$$

$$\to (x_0-x_2)(x_1-x_2)\Big[(x_0-\cancel{x_2}) - (x_1-\cancel{x_2})\Big]$$

$$\to (x_0-x_2)(x_1-x_2)(x_0-x_1)$$

<u>Kramer's rule:</u>

$$b = \frac{\begin{vmatrix} (x_0-x_2)^2 & f(x_0)-f(x_2) \\ (x_1-x_2)^2 & f(x_1)-f(x_2) \end{vmatrix}}{\begin{vmatrix} (x_0-x_2)^2 & x_0-x_2 \\ (x_1-x_2)^2 & x_1-x_2 \end{vmatrix} = (x_0-x_2)(x_1-x_2)(x_0-x_1)}$$

or

$$b = \frac{(x_0-x_2)^2\big(f(x_1)-f(x_2)\big) - (x_1-x_2)^2\big(f(x_0)-f(x_2)\big)}{(x_0-x_2)(x_1-x_2)(x_0-x_1)}$$

Finding X_3.

The new approximation X_3 is found by applying the quadratic formula to polynomial (2.2)., i.e,

$$z_3 = x_3 - x_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

An alternative way to write this equation is by

$$z_3 = \frac{1}{2a}\,\frac{\left[-b \pm \sqrt{b^2-4ac}\right]}{b \pm \sqrt{b^2-4ac}} \cdot \left[b \pm \sqrt{b^2-4ac}\right] =$$

$$= \frac{1}{2a}\,\frac{-\cancel{b^2} + \cancel{b^2} - 4ac}{b \pm \sqrt{b^2-4ac}} = \frac{-\cancel{4}^{2}ac}{\cancel{2}_{1}\cancel{a}\left[b \pm \sqrt{b^2-4ac}\right]} =$$

or

$$\boxed{z_3 = \frac{-2c}{b \pm \sqrt{b^2-4ac}}}$$

$$\Rightarrow \boxed{X_3 = X_2 + \frac{-2c}{b \pm \sqrt{b^2-4ac}}}.$$

two possibilities!

Rmk: Since the polynomial (2.2) has at most two roots. The criteria to choose between the two roots (in case they are different) is by selecting the closest to the previous approx.

For example, x_3 is selected such that

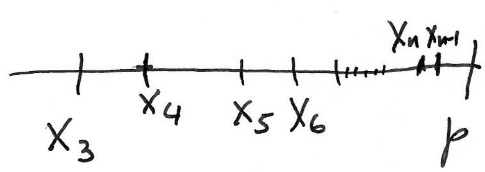
$|z_3| = |x_3 - x_2|$ is the smallest between the two possible roots.

This is equivalent to computing x_3 as

$$\boxed{z_3 = x_3 - x_2 = \frac{-2c}{E}}$$

where
$$E = \begin{cases} b + \sqrt{b^2-4ac}, & \text{if } |b+\sqrt{b^2-4ac}| > |b-\sqrt{b^2-4ac}| \\ \quad\text{or} & \\ b - \sqrt{b^2-4ac}, & \text{otherwise.} \end{cases}$$

If we keep doing this selection with the following approximations Müller's method will converge in general to one of the roots of $f(x)=0$. (polynomial of arbitrary degree)



While applying Muller's Method, the polynomial

$$\boxed{P(x) = a(x-x_2)^2 + b(x-x_2) + c} \qquad (2.1)$$

change its coefficients at every step

If the root that we are appraching to is Complex, then, most likely the coefficients of (2.1) will be Complex in most steps.

<u>Example</u> 3.- (book)

$$f(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$$

It has two complex and two real roots:

1) $-0.356 + 0.163\,i$ 2) $-0.356 - 0.163\,i$

3) 1.242 4) 1.970

If we start with the first three estimates:

I) $x_0 = 0.5, \quad x_1 = -0.5, \quad x_2 = 0$

$\Rightarrow a = 9, \quad b = 10, \quad c = 6$

$$P(x) = 9x^2 + 10x + 6.$$

New approx.: $x_3 = -0.556 + 0.598\,i$

II) Using $x_1, x_2, x_3 \Rightarrow a = 54.88 + 39.36\,i$

$b = 19.06 - 89.72\,i$

$c = -29.40 + 3.90\,i$

The reason why Müller's Method still work in the presence of complex coefficients is because the Quadratic equation for second order polynomial still works when the coefficients are complex.

See next page.